

Noncentral bimatrix variate generalised beta distributions

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Abstract

In this paper, we determine the density functions of nonsymmetrised doubly noncentral matrix variate beta type I and II distributions. The nonsymmetrised density functions of doubly noncentral and noncentral bimatrix variate generalised beta type I and II distributions are also obtained.

1 Introduction

When we consider generalising the distribution of a random variable to the multivariate case, two options are normally addressed, those of extending it to either the vectorial or the matrix cases, e.g. normal, t or bessel distributions, among many others. However, some of these generalisations have traditionally been made directly to the matrix case, where such a matrix is symmetric - this is the case of the chi-square and beta distributions, for which the corresponding multivariate distributions are the Wishart and matrix variate beta distributions, respectively. Nevertheless, these latter generalisations are inappropriate in some cases, because sometimes we might be interested in a vectorial version and not in a matrix version. For example, we are interested in a random vector in which each marginal is a random variable beta (type I or II). Libby and Novick (1982) proposed a multivariate (vector) beta distribution. Some applications to utility modelling and Bayesian analysis are also presented in Libby and Novick (1982) and Chen and Novick (1984), respectively. In particular Olkin and Liu (2003) proposed the following bivariate version. Observe that the following definition eliminates the hypothesis that the variables have a chi-squared distribution, assuming, instead, a gamma distribution.

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Let A , B and C be distributed as independent gamma random variables with parameters $\alpha = a, b, c$, respectively and $\delta = 1$ in the three cases (see eq. (4) in Section 2), and define

$$U_1 = \frac{A}{A+C}, \quad U_2 = \frac{B}{B+C}. \quad (1)$$

Clearly, U_1 and U_2 each have a beta type I distribution, $U_1 \sim \mathcal{BI}_1(a, c)$ and $U_2 \sim \mathcal{BI}_1(b, c)$, over $0 \leq u_1, u_2 \leq 1$. However, they are correlated, and then $(U_1, U_2)'$ has a bivariate generalised beta type I distribution over $0 \leq u_1, u_2 \leq 1$.

A similar result is obtained in the case of beta type II. Now, let us define

$$F_1 = \frac{A}{C}, \quad F_2 = \frac{B}{C}.$$

Once again it is easy to see that F_1 and F_2 each have a beta type II distribution, $F_1 \sim \mathcal{BII}_1(a, c)$ and $F_2 \sim \mathcal{BII}_1(b, c)$, over $f_1, f_2 \geq 0$. As in the beta type I case, they are correlated therefore and $(F_1, F_2)'$ has a bivariate generalised beta type II distribution over $f_1, f_2 \geq 0$.

These ideas can be extended to the matrix variate case. Thus, let us assume a partitioned matrix $\mathbb{U} = (\mathbf{U}_1; \mathbf{U}_2)' \in \mathbb{R}^{2m \times m}$, then under the matrix variate versions of the transformations (1), we are interested in finding the joint density of \mathbf{U}_1 and \mathbf{U}_2 , where it is easy to see that the marginal densities of \mathbf{U}_1 and \mathbf{U}_2 are matrix variate beta type I distributions. In the central case, the matrix variate joint densities of \mathbf{U}_1 and \mathbf{U}_2 and of \mathbf{F}_1 and \mathbf{F}_2 and some properties are studied in Díaz-García and Gutiérrez-Jáimez (2008). These distributions are termed central bimatrix variate generalised beta type I and II distributions, respectively. They play a potentially important role in the context of shape theory, specifically in affine or configuration densities, such as the Goodall and Mardia (1993) conjecture. Suppose that we have two samples of images of size n , each one of which is obtained at two times. Also, assume that we are interested in evaluating whether a learning process is present or whether the process has a memory. In this context, if we obtain as the configuration density a central, noncentral or double noncentral bimatrix variate generalised beta type I and II distribution, it might be possible to study these problems (learning or memory problems) and to compare the parameters of \mathbf{F}_1 (\mathbf{U}_1) and \mathbf{F}_2 (\mathbf{U}_2) considering the latter as bimatrix variate.

In this paper, we study bimatrix variate generalised beta type I and II distributions under different cases of noncentrality. Some definitions regarding the symmetrised function are given in Subsection 2.1 and Subsection 2.2 presents known and new results about central, noncentral and doubly noncentral matrix variate beta type I and II distributions; also we include the definition of the central bimatrix variate generalised beta type I and II distributions. Nonsymmetrised doubly noncentral density functions of the bimatrix variate generalised beta type I and II distributions are studied, and diverse noncentral cases of the bimatrix variate generalised beta type I and II distributions are obtained as particular cases of nonsymmetrised doubly noncentral density functions, see Sections 3 and 4, respectively.

2 Preliminary results

2.1 Symmetrised density function

In multivariate analysis there exist a large class of important hypothesis testing problems all of which may be tested by a set of criteria that depend functionally on the eigenvalues of a matrix variate. With the propose to investigate the non-null distributions of these criteria, Greenacre (1973) introduce the notion of a **symmetrised distribution** of a matrix variate, a notion which facilitates many proofs in such derivations.

Given a density function $f_{\mathbf{X}}(\mathbf{X})$, $\mathbf{X} \in \Re^{m \times m}$, $\mathbf{X} > \mathbf{0}$, Greenacre (1973) proposes the following definition

$$f_s(\mathbf{X}) = \int_{\mathcal{O}(m)} f_{\mathbf{X}}(\mathbf{H}\mathbf{X}\mathbf{H}') (d\mathbf{H}), \quad \mathbf{H} \in \mathcal{O}(m)$$

where $\mathcal{O}(m) = \{\mathbf{H} \in \Re^{m \times m} | \mathbf{H}'\mathbf{H} = \mathbf{H}\mathbf{H}' = \mathbf{I}_m\}$ and $(d\mathbf{H})$ denotes the normalised Haar measure on $\mathcal{O}(m)$, see Muirhead (1982, pp. 60 and 260). This function $f_s(\mathbf{X})$ is termed symmetrised density function of \mathbf{X} .

Our proposal is to apply this idea from Greenacre (1973) in an inverse way, i.e. well-known the explicit expression of the symmetrised density function of \mathbf{X}

$$f_s(\mathbf{X}) = \int_{\mathcal{O}(m)} f(\mathbf{H}\mathbf{X}\mathbf{H}') (d\mathbf{H}). \quad (2)$$

We wish to identify the density function $f(\mathbf{X})$. The density function obtained by applying the idea underlying (2) is termed the nonsymmetrised density function. Finally, note that the joint density function of the eigenvalues of \mathbf{X} can be found from $f_s(\mathbf{X})$ or $f(\mathbf{X})$, indifferently.

2.2 Matrix variate beta distributions

In general, matrix variate beta type I and II distributions are defined in terms of two matrices, say, \mathbf{A} and \mathbf{B} , which are independent and have Wishart distributions, see Olkin and Rubin (1964), Khatri (1970), Muirhead (1982), Farrell (1985), Cadet (1996), Gupta and Nagar (2000), Díaz-García and Gutiérrez-Jáimez (2007, 2006, 2008), among many others. The present paper generalises these results, assuming that \mathbf{A} and \mathbf{B} have matrix variate gamma distributions.

The $m \times m$ matrix \mathbf{A} is said to have a noncentral matrix variate gamma distribution with parameters $a \in \Re$ in which Θ is an $m \times m$ positive definite matrix and Ω is an $m \times m$ matrix, this fact being denoted as $\mathbf{A} \sim \mathcal{G}_m(a, \Theta, \Omega)$, if its density function is (see Muirhead (1982, pp. 57 and 61) and Gupta and Nagar (2000))

$$\mathcal{G}_m(\mathbf{A}; a, \Theta, \Omega) = \mathcal{G}_m(\mathbf{A}; a, \Theta) {}_0F_1(a, \Omega\Theta^{-1}\mathbf{A}), \quad \mathbf{A} > \mathbf{0}, \quad (3)$$

where ${}_0F_1(\cdot)$ is a hypergeometric function with a matrix argument (see Muirhead (1982, p. 258)) and $\mathcal{G}_m(\mathbf{A}; a, \Theta) \equiv \mathcal{G}_m(\mathbf{A}; a, \Theta, \mathbf{0})$ denotes the density function of a central matrix variate gamma distribution given by

$$\mathcal{G}_m(\mathbf{A}; a, \Theta) = \frac{|\mathbf{A}|^{a-(m+1)/2}}{\Gamma_m[a]|\Theta|^a} \text{etr}(-\Theta^{-1}\mathbf{A}), \quad \mathbf{A} > \mathbf{0}, \quad (4)$$

and denoted as $\mathbf{A} \sim \mathcal{G}_m(a, \Theta) \equiv \mathcal{G}_m(a, \Theta, \mathbf{0})$. Where $\text{etr}(\cdot) \equiv \exp(\text{tr}(\cdot))$ and $\Gamma_m[a]$ denotes the multivariate gamma function and is defined as

$$\Gamma_m[a] = \int_{\mathbf{V} > \mathbf{0}} \text{etr}(-\mathbf{V}) |\mathbf{V}|^{a-(m+1)/2} (d\mathbf{V}),$$

$\text{Re}(a) > (m-1)/2$.

In addition to the classification of beta distributions as beta type I and type II (see Gupta and Nagar (2000) and Srivastava and Khatri (1979)), two definitions have been proposed for each of these, see Olkin and Rubin (1964), Srivastava (1968), Díaz-García and Gutiérrez-Jáimez (2001) and James (1964). Let us focus initially on the beta type I distribution; if \mathbf{A} and \mathbf{B}

have a matrix variate gamma distribution, i.e. $\mathbf{A} \sim \mathcal{G}_m(a, \mathbf{I}_m)$ and $\mathbf{B} \sim \mathcal{G}_m(b, \mathbf{I}_m)$ independently, then the beta matrix \mathbf{U} can be defined as

$$\mathbf{U} = \begin{cases} (\mathbf{A} + \mathbf{B})^{-1/2} \mathbf{A} (\mathbf{A} + \mathbf{B})^{-1/2}', & \text{Definition 1 or,} \\ \mathbf{A}^{1/2} (\mathbf{A} + \mathbf{B})^{-1} (\mathbf{A}^{1/2})', & \text{Definition 2,} \end{cases} \quad (5)$$

where $\mathbf{C}^{1/2}(\mathbf{C}^{1/2})' = \mathbf{C}$ is a reasonable nonsingular factorization of \mathbf{C} , see Gupta and Nagar (2000), Srivastava and Khatri (1979) and Muirhead (1982). It is readily apparent that under definition 1 and 2 the density function is

$$\mathcal{BI}_m(\mathbf{U}; a, b) = \frac{1}{\beta_m[a, b]} |\mathbf{U}|^{a-(m+1)/2} |\mathbf{I}_m - \mathbf{U}|^{b-(m+1)/2} (d\mathbf{U}), \quad \mathbf{0} < \mathbf{U} < \mathbf{I}_m, \quad (6)$$

writing this fact as $\mathbf{U} \sim \mathcal{BI}_m(a, b)$, with $\text{Re}(a) > (m-1)/2$ and $\text{Re}(b) > (m-1)/2$; where $\beta_m[a, b]$ denotes the multivariate beta function defined by

$$\begin{aligned} \beta_m[b, a] &= \int_{\mathbf{0} < \mathbf{S} < \mathbf{I}_m} |\mathbf{S}|^{a-(m+1)/2} |\mathbf{I}_m - \mathbf{S}|^{b-(m+1)/2} (d\mathbf{S}) \\ &= \int_{\mathbf{R} > \mathbf{0}} |\mathbf{R}|^{a-(m+1)/2} |\mathbf{I}_m + \mathbf{R}|^{-(a+b)} (d\mathbf{R}) \\ &= \frac{\Gamma_m[a] \Gamma_m[b]}{\Gamma_m[a+b]}. \end{aligned}$$

A similar situation arises with the beta type II distribution, and thus we have the following two definitions:

$$\mathbf{F} = \begin{cases} \mathbf{B}^{-1/2} \mathbf{A} (\mathbf{B}^{-1/2})', & \text{Definition 1,} \\ \mathbf{A}^{1/2} \mathbf{B}^{-1} (\mathbf{A}^{1/2})', & \text{Definition 2,} \end{cases} \quad (7)$$

with the distribution being denoted as $\mathbf{F} \sim \mathcal{BII}_m(a, b)$. In this case, under definitions 1 and 2, the density function of \mathbf{F} is

$$\mathcal{BII}_m(\mathbf{F}; a, b) = \frac{1}{\beta_m[a, b]} |\mathbf{F}|^{a-(m+1)/2} |\mathbf{I}_m + \mathbf{F}|^{-(a+b)}, \quad \mathbf{F} > \mathbf{0}. \quad (8)$$

Díaz-García and Gutiérrez-Jáimez (2007, 2006) showed that in doubly noncentral and noncentral matrix variate beta type I and II distributions, the corresponding density functions are invariant under definitions 1 and 2. Therefore, henceforth we shall make no distinction between definitions 1 and 2.

When these ideas are extended to the doubly noncentral case, i.e. when $\mathbf{A} \sim \mathcal{G}_m(a, \mathbf{I}_m, \mathbf{\Omega}_1)$ and $\mathbf{B} \sim \mathcal{G}_m(b, \mathbf{I}_m, \mathbf{\Omega}_2)$, strictly speaking, we have not found the densities of the matrix variate beta type I and II distributions. Rather, for the case of the beta type II distribution, (Chikuse, 1980) found the distribution of $\tilde{\mathbf{V}} = \tilde{\mathbf{B}}^{-1/2} \tilde{\mathbf{A}} (\tilde{\mathbf{B}}^{-1/2})'$ where $\tilde{\mathbf{A}} = \mathbf{H}' \mathbf{A} \mathbf{H}$ and $\tilde{\mathbf{B}} = \mathbf{H}' \mathbf{B} \mathbf{H}$, $\mathbf{H} \in \mathcal{O}(m)$, with $\mathcal{O}(m) = \{\mathbf{H} \in \mathbb{R}^{m \times m} | \mathbf{H} \mathbf{H}' = \mathbf{H}' \mathbf{H} = \mathbf{I}_m\}$. It is straightforward to show that the procedure proposed by Chikuse (1980) and Chikuse and Davis (1986) is equivalent to finding the symmetrised density defined by Greenacre (1973), see also Roux (1975). From Díaz-García and Gutiérrez-Jáimez (2006) and using the notation for the operator sum as in Davis (1980) we have the following:

1. The symmetrised density function of doubly noncentral matrix variate beta type I is

$$\begin{aligned} &\mathcal{BI}_m(\mathbf{U}; a, b) \text{etr}(-(\mathbf{\Omega}_1 + \mathbf{\Omega}_2)) \\ &\times \sum_{\kappa, \lambda; \phi}^{\infty} \frac{(a+b)_{\phi}}{(a)_{\kappa} (b)_{\lambda} k! \ell!} \frac{C_{\phi}^{\kappa, \lambda}(\mathbf{\Omega}_1, \mathbf{\Omega}_2) C_{\phi}^{\kappa, \lambda}(\mathbf{U}, (\mathbf{I}_m - \mathbf{U}))}{C_{\phi}(\mathbf{I}_m)}, \quad \mathbf{0} < \mathbf{U} < \mathbf{I}_m. \end{aligned} \quad (9)$$

2. and the symmetrised density function of doubly noncentral matrix variate beta type II is

$$\begin{aligned} & \mathcal{B}II_m(\mathbf{F}; a, b) \text{etr}(-(\mathbf{\Omega}_1 + \mathbf{\Omega}_2)) \\ & \times \sum_{\kappa, \lambda; \phi}^{\infty} \frac{(a+b)_{\phi}}{(a)_{\kappa} (b)_{\lambda} k! l!} \frac{C_{\phi}^{\kappa, \lambda}(\mathbf{\Omega}_1, \mathbf{\Omega}_2) C_{\phi}^{\kappa, \lambda}((\mathbf{I}_m + \mathbf{F})^{-1} \mathbf{F}, (\mathbf{I}_m + \mathbf{F})^{-1})}{C_{\phi}(\mathbf{I}_m)}. \end{aligned} \quad (10)$$

where $\mathbf{F} > \mathbf{0}$, $\text{Re}(a) > (m-1)/2$, $\text{Re}(b) > (m-1)/2$, $(a)_{\tau}$ is the generalised hypergeometric coefficient or product of Pochhammer symbols and $C_{\phi}^{\kappa, \lambda}(\cdot, \cdot)$ denotes the invariant polynomials with the matrix arguments defined in Davis (1980), see also Chikuse (1980) and Chikuse and Davis (1986).

As particular cases of doubly noncentral distributions it is possible to obtain two different definitions of noncentral distributions, given another classification, in which the beta matrix is defined as follows, see Greenacre (1973) and Gupta and Nagar (2000):

$$\begin{aligned} \mathbf{W} &= \mathbf{A}^{1/2}(\mathbf{A} + \mathbf{B})^{-1}(\mathbf{A}^{1/2})', \text{ denoting as } \mathcal{B}I(A)_m(a, b, \mathbf{\Omega}_2), (\mathbf{\Omega}_1 = \mathbf{0}) \\ \mathbf{U} &= \mathbf{A}^{1/2}(\mathbf{A} + \mathbf{B})^{-1}(\mathbf{A}^{1/2})', \text{ denoting as } \mathcal{B}I(B)_m(a, b, \mathbf{\Omega}_1), (\mathbf{\Omega}_2 = \mathbf{0}). \end{aligned}$$

Similarly, in the case of beta type II we have

$$\begin{aligned} \mathbf{V} &= \mathbf{B}^{-1/2} \mathbf{A}(\mathbf{B}^{-1/2})', \text{ denoting as } \mathcal{B}II(A)_m(a, b, \mathbf{\Omega}_2), (\mathbf{\Omega}_1 = \mathbf{0}) \\ \mathbf{F} &= \mathbf{B}^{-1/2} \mathbf{A}(\mathbf{B}^{-1/2})', \text{ denoting as } \mathcal{B}II(B)_m(a, b, \mathbf{\Omega}_1), (\mathbf{\Omega}_2 = \mathbf{0}) \end{aligned}$$

Both distributions, types A and B, play a fundamental role in various areas of statistics, for example in the W and U criteria proposed by Díaz-García and Caro-Lopera (2008).

The symmetrised and nonsymmetrised density functions of \mathbf{W} , \mathbf{U} , \mathbf{V} and \mathbf{F} can be obtained as particular cases of (9) and (10). All these densities are found in Díaz-García and Gutiérrez-Jáimez (2007).

Now, using the approach described in Díaz-García and Gutiérrez-Jáimez (2007) we can find the (nonsymmetrised) density functions of doubly noncentral matrix variate beta type I and II distributions. Observe that for (9) and by Díaz-García (2006),

$$\int_{\mathcal{O}(m)} C_{\phi}^{\kappa, \lambda}(\mathbf{\Omega}_1 \mathbf{H} \mathbf{U} \mathbf{H}', \mathbf{\Omega}_2 (\mathbf{I}_m - \mathbf{H} \mathbf{U} \mathbf{H}')) (d\mathbf{H}) = \frac{C_{\phi}^{\kappa, \lambda}(\mathbf{\Omega}_1, \mathbf{\Omega}_2) C_{\phi}^{\kappa, \lambda}(\mathbf{U}, (\mathbf{I}_m - \mathbf{U}))}{\theta_{\phi}^{\kappa, \lambda} C_{\phi}(\mathbf{I}_m)}.$$

where $\theta_{\phi}^{\kappa, \lambda}$ is defined in Davis (1979) and Chikuse (1980).

Proceeding in analogous form for (10), we have the following.

Theorem 2.1. For $\text{Re}(a) > (m-1)/2$ and $\text{Re}(b) > (m-1)/2$,

1. the nonsymmetrised density function of the doubly noncentral matrix variate beta type I is

$$\mathcal{B}I_m(\mathbf{U}; a, b) \text{etr}(-(\mathbf{\Omega}_1 + \mathbf{\Omega}_2)) \quad (11)$$

$$\times \sum_{\kappa, \lambda; \phi}^{\infty} \frac{(a+b)_{\phi} \theta_{\phi}^{\kappa, \lambda}}{(a)_{\kappa} (b)_{\lambda} k! l!} C_{\phi}^{\kappa, \lambda}(\mathbf{\Omega}_1 \mathbf{U}, \mathbf{\Omega}_2 (\mathbf{I}_m - \mathbf{U})), \quad \mathbf{0} < \mathbf{U} < \mathbf{I}_m.$$

which is denoted as $\mathbf{U} \sim \mathcal{B}I_m(a, b, \mathbf{\Omega}_1, \mathbf{\Omega}_2)$.

2. and the nonsymmetrised density function of the doubly noncentral matrix variate beta type II is

$$\mathcal{B}II_m(\mathbf{F}; a, b) \text{etr}(-(\mathbf{\Omega}_1 + \mathbf{\Omega}_2)) \quad (12)$$

$$\times \sum_{\kappa, \lambda; \phi}^{\infty} \frac{(a+b)_{\phi} \theta_{\phi}^{\kappa, \lambda}}{(a)_{\kappa} (b)_{\lambda} k! l!} C_{\phi}^{\kappa, \lambda} (\mathbf{\Omega}_1 (\mathbf{I}_m + \mathbf{F})^{-1} \mathbf{F}, \mathbf{\Omega}_2 (\mathbf{I}_m + \mathbf{F})^{-1}), \quad \mathbf{F} > \mathbf{0},$$

which is denoted as $\mathbf{U} \sim \mathcal{BII}_m(a, b, \mathbf{\Omega}_1, \mathbf{\Omega}_2)$.

Doubly noncentral, noncentral and central matrix variate beta type I and II distributions play a very important role in diverse problems for proving hypotheses in the context of multivariate analysis, including canonical correlation analysis, the general linear hypothesis in MANOVA and multiple matrix variate correlation analysis, see Muirhead (1982), Rao (1973), Srivastava (1968) and Kshirsagar (1961). Similarly, doubly noncentral and noncentral beta distributions are to be found in the context of econometrics and shape theory, see Chikuse and Davis (1986) and Goodall and Mardia (1993), respectively.

Now from Díaz-García and Gutiérrez-Jáimez (2008); let \mathbf{A} , \mathbf{B} and \mathbf{C} be independent, where $\mathbf{A} \sim \mathcal{G}_m(a, \mathbf{I}_m)$, $\mathbf{B} \sim \mathcal{G}_m(b, \mathbf{I}_m)$ and $\mathbf{C} \sim \mathcal{G}_m(c, \mathbf{I}_m)$ with $\text{Re}(a) > (m-1)/2$, $\text{Re}(b) > (m-1)/2$ and $\text{Re}(c) > (m-1)/2$ and let us define

$$\mathbf{U}_1 = (\mathbf{A} + \mathbf{C})^{-1/2} \mathbf{A} (\mathbf{A} + \mathbf{C})^{-1/2} \quad \text{and} \quad \mathbf{U}_2 = (\mathbf{B} + \mathbf{C})^{-1/2} \mathbf{B} (\mathbf{B} + \mathbf{C})^{-1/2} \quad (13)$$

Of course, $\mathbf{U}_1 \sim \mathcal{BI}_m(a, c)$ and $\mathbf{U}_2 \sim \mathcal{BI}_m(b, c)$. However, they are correlated and therefore the distribution of $\mathbb{U} = (\mathbf{U}_1; \mathbf{U}_2)' \in \mathbb{R}^{2m \times m}$ is termed a central bimatrix variate generalised beta type I distribution, denoted as $\mathbb{U} \sim \mathcal{BGBI}_{2m \times m}(a, b, c)$. Moreover, its density function is

$$\frac{|\mathbf{U}_1|^{a-(m+1)/2} |\mathbf{U}_2|^{b-(m+1)/2} |\mathbf{I}_m - \mathbf{U}_1|^{b+c-(m+1)/2} |\mathbf{I}_m - \mathbf{U}_2|^{a+c-(m+1)/2}}{\beta_m^*[a, b, c] |\mathbf{I}_m - \mathbf{U}_1 \mathbf{U}_2|^{a+b+c}} \quad (14)$$

and is denoted as $\mathcal{BGBI}_{2m \times m}(\mathbb{U}; a, b, c)$, where $\mathbf{0} < \mathbf{U}_1 < \mathbf{I}_m$, $\mathbf{0} < \mathbf{U}_2 < \mathbf{I}_m$ with $\text{Re}(a) > (m-1)/2$, $\text{Re}(b) > (m-1)/2$ and $\text{Re}(c) > (m-1)/2$ and

$$\beta_m^*[a, b, c] = \frac{\Gamma_m[a] \Gamma_m[b] \Gamma_m[c]}{\Gamma_m[a+b+c]}$$

Similarly, let

$$\mathbf{F}_1 = \mathbf{C}^{-1/2} \mathbf{A} \mathbf{C}^{-1/2} \quad \text{and} \quad \mathbf{F}_2 = \mathbf{C}^{-1/2} \mathbf{B} \mathbf{C}^{-1/2} \quad (15)$$

Clearly, $\mathbf{F}_1 \sim \mathcal{BII}_m(a, c)$ and $\mathbf{F}_2 \sim \mathcal{BII}_m(b, c)$. But they are correlated and then the distribution of $\mathbb{F} = (\mathbf{F}_1; \mathbf{F}_2)' \in \mathbb{R}^{2m \times m}$ is termed a central bimatrix variate generalised beta type II distribution, which is denoted as $\mathbb{F} \sim \mathcal{BGBI}_{2m \times m}(a, b, c)$. Its density function is

$$\mathcal{BGBI}_{2m \times m}(\mathbb{F}; a, b, c) = \frac{|\mathbf{F}_1|^{a-(m+1)/2} |\mathbf{F}_2|^{b-(m+1)/2}}{\beta_m^*[a, b, c] |\mathbf{I}_m + \mathbf{F}_1 + \mathbf{F}_2|^{a+b+c}} \quad (16)$$

where $\mathbf{F}_1 > \mathbf{0}$, $\mathbf{F}_2 > \mathbf{0}$, with $\text{Re}(a) > (m-1)/2$, $\text{Re}(b) > (m-1)/2$ and $\text{Re}(c) > (m-1)/2$.

Other properties of bimatrix variate generalised beta type I and II distributions are studied in Díaz-García and Gutiérrez-Jáimez (2008).

The use of matrix and bimatrix variate beta-type distributions has not been developed as expected and hoped, due particularly to the fact that such distributions depend on hypergeometric functions with a matrix argument or on zonal polynomials, which until very recently were quite complicated to evaluate. Recently, descriptions have been made of algorithms that are very efficient at calculating both zonal polynomials and hypergeometric functions with a matrix argument; these can be used more widely and more efficiently in noncentral distributions in general, see Koev (2009) and Koev and Edelman (2006).

3 Doubly noncentral bimatrix variate generalised beta type I distribution

In this section we derive the doubly noncentral bimatrix variate generalised beta type I distribution.

Theorem 3.1. Let \mathbf{A} , \mathbf{B} and \mathbf{C} be independent random matrices, such that $\mathbf{A} \sim \mathcal{G}_m(a, \mathbf{I}_m, \boldsymbol{\Omega}_1)$, $\mathbf{B} \sim \mathcal{G}_m(b, \mathbf{I}_m, \boldsymbol{\Omega}_2)$ and $\mathbf{C} \sim \mathcal{G}_m(c, \mathbf{I}_m, \boldsymbol{\Omega}_3)$ with $\text{Re}(a) > (m-1)/2$, $\text{Re}(b) > (m-1)/2$ and $\text{Re}(c) > (m-1)/2$ and let us define

$$\mathbf{U}_1 = (\mathbf{A} + \mathbf{C})^{-1/2} \mathbf{A} (\mathbf{A} + \mathbf{C})^{-1/2} \quad \text{and} \quad \mathbf{U}_2 = (\mathbf{B} + \mathbf{C})^{-1/2} \mathbf{B} (\mathbf{B} + \mathbf{C})^{-1/2} \quad (17)$$

Then the symmetrised density function of $\mathbb{U} = (\mathbf{U}_1; \mathbf{U}_2)' \in \mathbb{R}^{2m \times m}$ is

$$\begin{aligned} \mathcal{BGBI}_{2m \times m}(\mathbb{U}; a, b, c) \text{etr}\{-(\boldsymbol{\Omega}_1 + \boldsymbol{\Omega}_2 + \boldsymbol{\Omega}_3)\} & \sum_{\kappa, \tau, \lambda; \phi}^{\infty} \frac{(a+b+c)_{\phi}}{(a)_{\kappa}(b)_{\tau}(c)_{\lambda} k! t! l!} \\ & \times \frac{C_{\phi}^{\kappa, \tau, \lambda}(\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2, \boldsymbol{\Omega}_3) C_{\phi}^{\kappa, \tau, \lambda}(\mathbf{M}_1, \mathbf{M}_2, \mathbf{M})}{C_{\phi}(\mathbf{I}_m)}, \end{aligned} \quad (18)$$

and then the nonsymmetrised density function of $\mathbb{U} = (\mathbf{U}_1; \mathbf{U}_2)' \in \mathbb{R}^{2m \times m}$ is

$$\begin{aligned} \mathcal{BGBI}_{2m \times m}(\mathbb{U}; a, b, c) \text{etr}\{-(\boldsymbol{\Omega}_1 + \boldsymbol{\Omega}_2 + \boldsymbol{\Omega}_3)\} & \sum_{\kappa, \tau, \lambda; \phi}^{\infty} \frac{(a+b+c)_{\phi} \theta_{\phi}^{\kappa, \tau, \lambda}}{(a)_{\kappa}(b)_{\tau}(c)_{\lambda} k! t! l!} \\ & \times C_{\phi}^{\kappa, \tau, \lambda}(\boldsymbol{\Omega}_1 \mathbf{M}_1, \boldsymbol{\Omega}_2 \mathbf{M}_2, \boldsymbol{\Omega}_3 \mathbf{M}), \end{aligned} \quad (19)$$

which is denoted as $\mathbb{U} \sim \mathcal{BGBI}_{2m \times m}(a, b, c, \boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2, \boldsymbol{\Omega}_3)$; where $\mathbf{0} < \mathbf{U}_1 < \mathbf{I}_m$ and $\mathbf{0} < \mathbf{U}_2 < \mathbf{I}_m$ and

$$\begin{aligned} \mathbf{M}_1 &= (\mathbf{I}_m - \mathbf{U}_2)(\mathbf{I}_m - \mathbf{U}_1 \mathbf{U}_2)^{-1} \mathbf{U}_1, \\ \mathbf{M}_2 &= (\mathbf{I}_m - \mathbf{U}_1)(\mathbf{I}_m - \mathbf{U}_1 \mathbf{U}_2)^{-1} \mathbf{U}_2, \\ \mathbf{M} &= (\mathbf{I}_m - \mathbf{U}_1)(\mathbf{I}_m - \mathbf{U}_1 \mathbf{U}_2)^{-1} (\mathbf{I}_m - \mathbf{U}_2). \end{aligned}$$

Proof. The joint density function of \mathbf{A} , \mathbf{B} and \mathbf{C} is

$$\begin{aligned} & \frac{\text{etr}\{-(\boldsymbol{\Omega}_1 + \boldsymbol{\Omega}_2 + \boldsymbol{\Omega}_3)\}}{\Gamma_m[a] \Gamma_m[b] \Gamma_m[c]} |\mathbf{A}|^{a-(m+1)/2} |\mathbf{B}|^{b-(m+1)/2} |\mathbf{C}|^{c-(m+1)/2} \text{etr}\{-(\mathbf{A} + \mathbf{B} + \mathbf{C})\} \\ & \times {}_0F_1(a; \boldsymbol{\Omega}_1 \mathbf{A}) {}_0F_1(b; \boldsymbol{\Omega}_2 \mathbf{B}) {}_0F_1(c; \boldsymbol{\Omega}_3 \mathbf{C}), \end{aligned}$$

Now, consider the transformations (13) and $\mathbf{C} = \mathbf{C}$, then

$$(d\mathbf{A})(d\mathbf{B})(d\mathbf{C}) = |\mathbf{C}|^{m+1} |\mathbf{I}_m - \mathbf{U}_1|^{-(m+1)} |\mathbf{I}_m - \mathbf{U}_2|^{-(m+1)} (d\mathbf{U}_1)(d\mathbf{U}_2)(d\mathbf{C}).$$

The joint density function of \mathbf{U}_1 , \mathbf{U}_2 and \mathbf{C} is

$$\begin{aligned} & \frac{\text{etr}\{-(\boldsymbol{\Omega}_1 + \boldsymbol{\Omega}_2 + \boldsymbol{\Omega}_3)\}}{\Gamma_m[a] \Gamma_m[b] \Gamma_m[c]} \frac{|\mathbf{U}_1|^{a-(m+1)/2} |\mathbf{U}_2|^{b-(m+1)/2}}{|\mathbf{I}_m - \mathbf{U}_1|^{a+(m+1)/2} |\mathbf{I}_m - \mathbf{U}_2|^{b+(m+1)/2}} \\ & \times |\mathbf{C}|^{a+b+c-(m+1)/2} \exp\{-(\mathbf{I}_m - \mathbf{U}_1)^{-1} (\mathbf{I}_m - \mathbf{U}_1 \mathbf{U}_2) (\mathbf{I}_m - \mathbf{U}_2)^{-1} \mathbf{C}\}, \\ & \times {}_0F_1\left(a; \boldsymbol{\Omega}_1 \mathbf{C}^{1/2} (\mathbf{I}_m - \mathbf{U}_1)^{-1} \mathbf{U}_1 \mathbf{C}^{1/2}\right) {}_0F_1\left(b; \boldsymbol{\Omega}_2 \mathbf{C}^{1/2} (\mathbf{I}_m - \mathbf{U}_2)^{-1} \mathbf{U}_2 \mathbf{C}^{1/2}\right), \\ & \times {}_0F_1(c; \boldsymbol{\Omega}_3 \mathbf{C}). \end{aligned} \quad (20)$$

Now, note that

i)

$$(\mathbf{I}_m - \mathbf{U}_1)^{-1} - \mathbf{I}_m = \begin{cases} (\mathbf{I}_m - \mathbf{U}_1)^{-1} \mathbf{U}_1, \\ \mathbf{U}_1 (\mathbf{I}_m - \mathbf{U}_1)^{-1}. \end{cases}$$

With similarly expressions for $(\mathbf{I}_m - \mathbf{U}_2)^{-1} - \mathbf{I}_m$.

ii) From the argument of $\text{etr}(\cdot)$ in (20), denotes $\mathbf{M}^{-1} = (\mathbf{I}_m - \mathbf{U}_1)^{-1} \mathbf{U}_1 + (\mathbf{I}_m - \mathbf{U}_2)^{-1} \mathbf{U}_2 + \mathbf{I}_m$ and note that

$$\mathbf{M}^{-1} = \begin{cases} (\mathbf{I}_m - \mathbf{U}_1)^{-1} (\mathbf{I}_m - \mathbf{U}_1 \mathbf{U}_2) (\mathbf{I}_m - \mathbf{U}_2)^{-1}, \\ (\mathbf{I}_m - \mathbf{U}_2)^{-1} (\mathbf{I}_m - \mathbf{U}_1 \mathbf{U}_2) (\mathbf{I}_m - \mathbf{U}_1)^{-1}. \end{cases}$$

iii) Now assuming that $(\mathbf{I}_m - \mathbf{U}_1)^{-1} \mathbf{U}_1 \mathbf{M}$ is an argument of a symmetric function ($f(\mathbf{A}\mathbf{B}) = f(\mathbf{B}\mathbf{A})$) by i) and ii) we have

$$(\mathbf{I}_m - \mathbf{U}_1)^{-1} \mathbf{U}_1 \mathbf{M} = \begin{cases} (\mathbf{I}_m - \mathbf{U}_2) (\mathbf{I}_m - \mathbf{U}_1 \mathbf{U}_2)^{-1} \mathbf{U}_1 \\ (\mathbf{I}_m - \mathbf{U}_1 \mathbf{U}_2)^{-1} (\mathbf{I}_m - \mathbf{U}_2) \mathbf{U}_1, \\ \mathbf{U}_1 (\mathbf{I}_m - \mathbf{U}_2) (\mathbf{I}_m - \mathbf{U}_1 \mathbf{U}_2)^{-1}, \\ \mathbf{U}_1 (\mathbf{I}_m - \mathbf{U}_1 \mathbf{U}_2)^{-1} (\mathbf{I}_m - \mathbf{U}_2). \end{cases}$$

With similar expressions for $(\mathbf{I}_m - \mathbf{U}_2)^{-1} \mathbf{U}_2 \mathbf{M}$.

The marginal joint density function of \mathbf{U}_1 and \mathbf{U}_2 is

$$\begin{aligned} & \frac{\text{etr}\{-(\boldsymbol{\Omega}_1 + \boldsymbol{\Omega}_2 + \boldsymbol{\Omega}_3)\} |\mathbf{U}_1|^{a-(m+1)/2} |\mathbf{U}_2|^{b-(m+1)/2}}{\Gamma_m[a] \Gamma_m[b] \Gamma_m[c] |\mathbf{I}_m - \mathbf{U}_1|^{a+(m+1)/2} |\mathbf{I}_m - \mathbf{U}_2|^{b+(m+1)/2}} \\ & \times \int_{\mathbf{C} > \mathbf{0}} |\mathbf{C}|^{a+b+c-(m+1)/2} \exp\{-\mathbf{M}^{-1} \mathbf{C}\} {}_0F_1\left(a; \boldsymbol{\Omega}_1 \mathbf{C}^{1/2} (\mathbf{I}_m - \mathbf{U}_1)^{-1} \mathbf{U}_1 \mathbf{C}^{1/2}\right), \\ & \times {}_0F_1\left(b; \boldsymbol{\Omega}_2 \mathbf{C}^{1/2} (\mathbf{I}_m - \mathbf{U}_2)^{-1} \mathbf{U}_2 \mathbf{C}^{1/2}\right) {}_0F_1(c; \boldsymbol{\Omega}_3 \mathbf{C}) (d\mathbf{C}), \end{aligned}$$

Denoting the density joint function of \mathbf{U}_1 and \mathbf{U}_2 as $f_{\mathbf{U}_1, \mathbf{U}_2}(\mathbf{U}_1, \mathbf{U}_2)$, considering the corresponding symmetrised function

$$f_s(\mathbf{U}_1, \mathbf{U}_2) = \int_{\mathcal{O}(m)} f_{\mathbf{U}_1, \mathbf{U}_2}(\mathbf{H} \mathbf{U}_1 \mathbf{H}', \mathbf{H} \mathbf{U}_2 \mathbf{H}') (d\mathbf{H})$$

and the transformation $\mathbf{C} = \mathbf{H} \mathbf{C} \mathbf{H}'$ with $(d\mathbf{C}) = (d(\mathbf{H} \mathbf{C} \mathbf{H}'))$. Then expanding the hypergeometric functions ${}_0F_1(\cdot)$ in terms of zonal polynomials we obtain that $f_s(\mathbf{U}_1, \mathbf{U}_2)$ is

$$\begin{aligned} & \frac{\text{etr}\{-(\boldsymbol{\Omega}_1 + \boldsymbol{\Omega}_2 + \boldsymbol{\Omega}_3)\} |\mathbf{U}_1|^{a-(m+1)/2} |\mathbf{U}_2|^{b-(m+1)/2}}{\Gamma_m[a] \Gamma_m[b] \Gamma_m[c] |\mathbf{I}_m - \mathbf{U}_1|^{a+(m+1)/2} |\mathbf{I}_m - \mathbf{U}_2|^{b+(m+1)/2}} \\ & \times \sum_{k=0}^{\infty} \sum_{t=0}^{\infty} \sum_{l=0}^{\infty} \sum_{\kappa} \sum_{\tau} \sum_{\lambda} \frac{1}{(a)_{\kappa} (b)_{\tau} (c)_{\lambda} k! t! l!} \int_{\mathbf{C} > \mathbf{0}} |\mathbf{C}|^{a+b+c-(m+1)/2} \exp\{-\mathbf{M}^{-1} \mathbf{C}\}, \\ & \times \left[\int_{\mathcal{O}(m)} C_{\kappa} \left(\boldsymbol{\Omega}_1 \mathbf{H} \mathbf{C}^{1/2} (\mathbf{I}_m - \mathbf{U}_1)^{-1} \mathbf{U}_1 \mathbf{C}^{1/2} \mathbf{H}' \right), \right. \\ & \times \left. C_{\tau} \left(\boldsymbol{\Omega}_2 \mathbf{H} \mathbf{C}^{1/2} (\mathbf{I}_m - \mathbf{U}_2)^{-1} \mathbf{U}_2 \mathbf{C}^{1/2} \mathbf{H}' \right) C_{\lambda} \left(\boldsymbol{\Omega}_3 \mathbf{H} \mathbf{C} \mathbf{H}' \right) (d\mathbf{H}) \right] (d\mathbf{C}), \end{aligned}$$

By integrating with respect to \mathbf{H} , using Chikuse and Davis (1986, equation (2.2)) and the notation for the operator sum as in Davis (1980), we have

$$\frac{\text{etr}\{-(\boldsymbol{\Omega}_1 + \boldsymbol{\Omega}_2 + \boldsymbol{\Omega}_3)\} |\mathbf{U}_1|^{a-(m+1)/2} |\mathbf{U}_2|^{b-(m+1)/2}}{\Gamma_m[a] \Gamma_m[b] \Gamma_m[c] |\mathbf{I}_m - \mathbf{U}_1|^{a+(m+1)/2} |\mathbf{I}_m - \mathbf{U}_2|^{b+(m+1)/2}}$$

$$\begin{aligned} & \times \sum_{\kappa, \tau, \lambda; \phi}^{\infty} \frac{C_{\phi}^{\kappa, \tau, \lambda}(\mathbf{\Omega}_1, \mathbf{\Omega}_2, \mathbf{\Omega}_3)}{(a)_{\kappa}(b)_{\tau}(c)_{\lambda} k! t! l!} \int_{\mathbf{C} > \mathbf{0}} |\mathbf{C}|^{a+b+c-(m+1)/2} \exp\{-\mathbf{M}^{-1}\mathbf{C}\}, \\ & \times \frac{C_{\phi}^{\kappa, \tau, \lambda}((\mathbf{I}_m - \mathbf{U}_1)^{-1}\mathbf{U}_1\mathbf{M}^{-1}, (\mathbf{I}_m - \mathbf{U}_2)^{-1}\mathbf{U}_2\mathbf{M}^{-1}, \mathbf{M}^{-1})}{C_{\phi}(\mathbf{I}_m)}(d\mathbf{C}). \end{aligned}$$

Finality, by integrating with respect to \mathbf{C} , see Chikuse (1980, equation (3.21)) and **iii**), we obtain the joint symmetrised density function of \mathbf{U}_1 and \mathbf{U}_2 .

The joint nonsymmetrised density function of \mathbf{U}_1 and \mathbf{U}_2 is obtained by applying the idea of Greenacre (1973) in an inverse way. With this proposal observe that $\mathcal{BGBI}_{2m \times m}(\mathbf{H}\mathbf{U}\mathbf{H}'; a, b, c) = \mathcal{BGBI}_{2m \times m}(\mathbb{U}; a, b, c)$ and by Díaz-García (2006)

$$\begin{aligned} & \int_{\mathcal{O}(m)} C_{\phi}^{\kappa, \tau, \lambda}(\mathbf{\Omega}_1 \mathbf{H} \mathbf{M}_1 \mathbf{H}', \mathbf{\Omega}_1 \mathbf{H} \mathbf{M}_2 \mathbf{H}', \mathbf{\Omega}_1 \mathbf{H} \mathbf{M} \mathbf{H}')(d\mathbf{H}) \\ & = \frac{C_{\phi}^{\kappa, \tau, \lambda}(\mathbf{\Omega}_1, \mathbf{\Omega}_2, \mathbf{\Omega}_3) C_{\phi}^{\kappa, \tau, \lambda}(\mathbf{M}_1, \mathbf{M}_2, \mathbf{M})}{\theta_{\phi}^{\kappa, \tau, \lambda} C_{\phi}(\mathbf{I}_m)}, \end{aligned}$$

from where the desired result is obtained. \square

In addition, note that in Theorem 3.1,

$$\mathbf{U}_1 \sim \mathcal{BI}_m(a, c, \mathbf{\Omega}_1, \mathbf{\Omega}_3) \quad \text{and} \quad \mathbf{U}_2 \sim \mathcal{BI}_m(b, c, \mathbf{\Omega}_2, \mathbf{\Omega}_3).$$

Next, assuming that one and/or two of the matrices \mathbf{A} , \mathbf{B} or \mathbf{C} have a central matrix variate gamma distribution in Theorem 3.1, let us study all the possible nonsymmetrised noncentral densities.

Corollary 3.1. *Let us assume the hypothesis of Theorem 3.1. Then the joint nonsymmetrised density function of \mathbf{U}_1 and \mathbf{U}_2 is:*

1. If $\mathbf{\Omega}_1 = \mathbf{\Omega}_2 = \mathbf{0}$

$$\frac{\mathcal{BGBI}_{2m \times m}(\mathbb{U}; a, b, c)}{\text{etr}\{\mathbf{\Omega}_3\}} {}_1F_1(a + b + c; c; \mathbf{\Omega}_3 \mathbf{M})$$

and $\mathbf{U}_1 \sim \mathcal{BI}(A)_m(a, c, \mathbf{\Omega}_3)$ and $\mathbf{U}_2 \sim \mathcal{BI}(A)_m(b, c, \mathbf{\Omega}_3)$.

2. If $\mathbf{\Omega}_3 = \mathbf{0}$

$$\frac{\mathcal{BGBI}_{2m \times m}(\mathbb{U}; a, b, c)}{\text{etr}\{\mathbf{\Omega}_1 + \mathbf{\Omega}_2\}} \sum_{\kappa, \tau; \phi}^{\infty} \frac{(a + b + c)_{\phi} \theta_{\phi}^{\kappa, \tau}}{(a)_{\kappa}(b)_{\tau} k! t!} C_{\phi}^{\kappa, \tau}(\mathbf{\Omega}_1 \mathbf{M}_1, \mathbf{\Omega}_2 \mathbf{M}_2)$$

also we have that $\mathbf{U}_1 \sim \mathcal{BI}(B)_m(a, c, \mathbf{\Omega}_1)$ and $\mathbf{U}_2 \sim \mathcal{BI}(B)_m(b, c, \mathbf{\Omega}_2)$.

3. If $\mathbf{\Omega}_2 = \mathbf{\Omega}_3 = \mathbf{0}$

$$\frac{\mathcal{BGBI}_{2m \times m}(\mathbb{U}; a, b, c)}{\text{etr}\{\mathbf{\Omega}_1\}} {}_1F_1(a + b + c; c; \mathbf{\Omega}_1 \mathbf{M}_1)$$

and with $\mathbf{U}_1 \sim \mathcal{BI}(B)_m(a, c, \mathbf{\Omega}_1)$ and $\mathbf{U}_2 \sim \mathcal{BI}_m(b, c)$.

4. If $\mathbf{\Omega}_1 = \mathbf{\Omega}_3 = \mathbf{0}$

$$\frac{\mathcal{BGBI}_{2m \times m}(\mathbb{U}; a, b, c)}{\text{etr}\{\mathbf{\Omega}_2\}} {}_1F_1(a + b + c; c; \mathbf{\Omega}_2 \mathbf{M}_2)$$

and $\mathbf{U}_1 \sim \mathcal{BI}_m(a, c)$ and $\mathbf{U}_2 \sim \mathcal{BI}(B)_m(b, c, \mathbf{\Omega}_2)$.

5. If $\Omega_2 = \mathbf{0}$

$$\frac{\mathcal{BGBI}_{2m \times m}(\mathbb{U}; a, b, c)}{\text{etr}\{\Omega_1 + \Omega_3\}} \sum_{\kappa, \lambda; \phi}^{\infty} \frac{(a+b+c)_{\phi} \theta_{\phi}^{\kappa, \lambda}}{(a)_{\kappa}(c)_{\lambda} k! l!} C_{\phi}^{\kappa, \lambda}(\Omega_1 \mathbf{M}_1, \Omega_3 \mathbf{M}).$$

In addition, note that $\mathbf{U}_1 \sim \mathcal{BI}_m(a, c, \Omega_1, \Omega_3)$ and $\mathbf{U}_2 \sim \mathcal{BI}(A)_m(b, c, \Omega_3)$.

6. If $\Omega_1 = \mathbf{0}$

$$\frac{\mathcal{BGBI}_{2m \times m}(\mathbb{U}; a, b, c)}{\text{etr}\{\Omega_2 + \Omega_3\}} \sum_{\tau, \lambda; \phi}^{\infty} \frac{(a+b+c)_{\phi} \theta_{\phi}^{\tau, \lambda}}{(b)_{\tau}(c)_{\lambda} t! l!} C_{\phi}^{\tau, \lambda}(\Omega_2 \mathbf{M}_2, \Omega_3 \mathbf{M})$$

and where $\mathbf{U}_1 \sim \mathcal{BI}(A)_m(a, c, \Omega_3)$ and $\mathbf{U}_2 \sim \mathcal{BI}_m(b, c, \Omega_2, \Omega_3)$.

Proof. The joint density functions of \mathbf{U}_1 and \mathbf{U}_2 in all items are a consequence of the basic properties of invariant polynomials, see Davis (1979, equations (2.1) and (2.3)), see also Chikuse (1980, equations (3.3) and (3.6)). The second claim in each of the items is a consequence of construction (17). \square

4 Doubly noncentral bimatrix variate generalised beta type II distribution

In this section we derive the doubly noncentral and noncentral bimatrix variate generalised beta type II distributions.

Theorem 4.1. Let \mathbf{A} , \mathbf{B} and \mathbf{C} be independent random matrices, such that $\mathbf{A} \sim \mathcal{G}_m(a, \mathbf{I}_m, \Omega_1)$, $\mathbf{B} \sim \mathcal{G}_m(b, \mathbf{I}_m, \Omega_2)$ and $\mathbf{C} \sim \mathcal{G}_m(c, \mathbf{I}_m, \Omega_3)$ with $\text{Re}(a) > (m-1)/2$, $\text{Re}(b) > (m-1)/2$ and $\text{Re}(c) > (m-1)/2$ and let us define

$$\mathbf{F}_1 = \mathbf{C}^{-1/2} \mathbf{A} \mathbf{C}^{-1/2} \quad \text{and} \quad \mathbf{F}_2 = \mathbf{C}^{-1/2} \mathbf{B} \mathbf{C}^{-1/2} \quad (21)$$

Then the symmetrised density function of $\mathbb{F} = (\mathbf{F}_1; \mathbf{F}_2)' \in \mathbb{R}^{2m \times m}$ is

$$\begin{aligned} \mathcal{BGBII}_{2m \times m}(\mathbb{F}; a, b, c) \text{etr}\{-(\Omega_1 + \Omega_2 + \Omega_3)\} & \sum_{\kappa, \tau, \lambda; \phi}^{\infty} \frac{(a+b+c)_{\phi}}{(a)_{\kappa}(b)_{\tau}(c)_{\lambda} k! t! l!} \\ & \times \frac{C_{\phi}^{\kappa, \tau, \lambda}(\Omega_1, \Omega_2, \Omega_3) C_{\phi}^{\kappa, \tau, \lambda}(\mathbf{N}_1, \mathbf{N}_2, \mathbf{N})}{C_{\phi}(\mathbf{I}_m)}, \end{aligned} \quad (22)$$

and then the nonsymmetrised density function of $\mathbb{F} = (\mathbf{F}_1; \mathbf{F}_2)' \in \mathbb{R}^{2m \times m}$ is

$$\begin{aligned} \mathcal{BGBII}_{2m \times m}(\mathbb{F}; a, b, c) \text{etr}\{-(\Omega_1 + \Omega_2 + \Omega_3)\} & \sum_{\kappa, \tau, \lambda; \phi}^{\infty} \frac{(a+b+c)_{\phi} \theta_{\phi}^{\kappa, \tau, \lambda}}{(a)_{\kappa}(b)_{\tau}(c)_{\lambda} k! t! l!} \\ & \times C_{\phi}^{\kappa, \tau, \lambda}(\Omega_1 \mathbf{N}_1, \Omega_2 \mathbf{N}_2, \Omega_3 \mathbf{N}), \end{aligned} \quad (23)$$

which is denoted as $\mathbb{F} \sim \mathcal{BGBII}_{2m \times m}(a, b, c, \Omega_1, \Omega_2, \Omega_3)$; where $\mathbf{F}_1 > \mathbf{0}$ and $\mathbf{F}_2 > \mathbf{0}$ and

$$\begin{aligned} \mathbf{N}_1 &= (\mathbf{I}_m + \mathbf{F}_1)^{-1} (\mathbf{I}_m + \mathbf{F}_2) (\mathbf{I}_m + \mathbf{F}_1 + \mathbf{F}_2)^{-1} \mathbf{F}_1, \\ \mathbf{N}_2 &= (\mathbf{I}_m + \mathbf{F}_1 + \mathbf{F}_2)^{-1} (\mathbf{I}_m + \mathbf{F}_1) (\mathbf{I}_m + \mathbf{F}_2)^{-1} \mathbf{F}_2, \\ \mathbf{N} &= (\mathbf{I}_m + \mathbf{F}_1 + \mathbf{F}_2)^{-1}. \end{aligned}$$

Proof. The proof is similar to that given for Theorem 4.1. Or alternatively, by noting that if $\mathbf{U} \sim \mathcal{B}I_m(a, b)$, then $(\mathbf{I}_m - \mathbf{U})^{-1} - \mathbf{I}_m \sim \mathcal{B}II_m(a, b)$, see Srivastava and Khatri (1979) and Díaz-García and Gutiérrez-Jáimez (2007). Then, by construction (17) we have that if $\mathbb{U} \sim \mathcal{BGB}II_{2m \times m}(a, b, c, \mathbf{\Omega}_1, \mathbf{\Omega}_2, \mathbf{\Omega}_3)$ then $\mathbb{F} = (\mathbf{F}_1 | \mathbf{F}_2)' \sim \mathcal{BGB}II_{2m \times m}(a, b, c, \mathbf{\Omega}_1, \mathbf{\Omega}_2, \mathbf{\Omega}_3)$, where

$$\mathbb{F} = \begin{pmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \end{pmatrix} = \begin{pmatrix} (\mathbf{I}_m - \mathbf{U}_1)^{-1} - \mathbf{I}_m \\ (\mathbf{I}_m - \mathbf{U}_2)^{-1} - \mathbf{I}_m \end{pmatrix}.$$

Then the inverse transformation is given by

$$\mathbb{U} = \begin{pmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{I}_m - (\mathbf{I}_m + \mathbf{F}_1)^{-1} \\ \mathbf{I}_m - (\mathbf{I}_m + \mathbf{F}_2)^{-1} \end{pmatrix}, \quad (24)$$

and the Jacobian is given by

$$(d\mathbf{U}_1)(d\mathbf{U}_2) = |\mathbf{I}_m + \mathbf{F}_1|^{-(m+1)} |\mathbf{I}_m + \mathbf{F}_2|^{-(m+1)} (d\mathbf{F}_1)(d\mathbf{F}_2).$$

Then, (22) follows, observing that under transformation (24)

i)

$$\frac{\mathcal{BGB}II_{2m \times m}((\mathbf{I}_m + (\mathbf{I}_m + \mathbf{F}_1)^{-1})(\mathbf{I}_m - (\mathbf{I}_m + \mathbf{F}_2)^{-1})'; a, b, c)}{|\mathbf{I}_m - \mathbf{F}_1|^{(m+1)} |\mathbf{I}_m + \mathbf{F}_2|^{(m+1)}} = \mathcal{BGB}II_{2m \times m}(\mathbb{F}; a, b, c),$$

ii) and

$$\begin{aligned} \mathbf{M}_1 &= (\mathbf{I}_m + \mathbf{F}_1)^{-1} (\mathbf{I}_m + \mathbf{F}_2) (\mathbf{I}_m + \mathbf{F}_1 + \mathbf{F}_2)^{-1} \mathbf{F}_1 = \mathbf{N}_1, \\ \mathbf{M}_2 &= (\mathbf{I}_m + \mathbf{F}_1 + \mathbf{F}_2)^{-1} (\mathbf{I}_m + \mathbf{F}_1) (\mathbf{I}_m + \mathbf{F}_2)^{-1} \mathbf{F}_2 = \mathbf{N}_2, \\ \mathbf{M} &= (\mathbf{I}_m + \mathbf{F}_1 + \mathbf{F}_2)^{-1} = \mathbf{N}. \end{aligned}$$

To obtain (23), observe that

$$\mathcal{BGB}II_{2m \times m}(\mathbf{H}\mathbb{F}\mathbf{H}'; a, b, c) = \mathcal{BGB}II_{2m \times m}(\mathbb{F}; a, b, c)$$

and by Díaz-García (2006)

$$\begin{aligned} \int_{\mathcal{O}(m)} C_{\phi}^{\kappa, \tau, \lambda}(\mathbf{\Omega}_1 \mathbf{H} \mathbf{N}_1 \mathbf{H}', \mathbf{\Omega}_1 \mathbf{H} \mathbf{N}_2 \mathbf{H}', \mathbf{\Omega}_1 \mathbf{H} \mathbf{N} \mathbf{H}') (d\mathbf{H}) \\ = \frac{C_{\phi}^{\kappa, \tau, \lambda}(\mathbf{\Omega}_1, \mathbf{\Omega}_2, \mathbf{\Omega}_3) C_{\phi}^{\kappa, \tau, \lambda}(\mathbf{N}_1, \mathbf{N}_2, \mathbf{N})}{\theta_{\phi}^{\kappa, \tau, \lambda} C_{\phi}^{\kappa, \tau, \lambda}(\mathbf{I}_m)}, \end{aligned}$$

from where the nonsymmetrised joint density function of \mathbf{F}_1 and \mathbf{F}_2 is obtained. \square

Finally, let us find the different possibilities of the nonsymmetrised noncentral density functions of bimatrix variate generalised beta type II distributions.

Corollary 4.1. *Under the hypothesis of Theorem 4.1 the joint nonsymmetrised density function of \mathbf{F}_1 and \mathbf{F}_2 is:*

1. If $\mathbf{\Omega}_1 = \mathbf{\Omega}_2 = \mathbf{0}$

$$\frac{\mathcal{BGB}II_{2m \times m}(\mathbb{F}; a, b, c)}{\text{etr}\{\mathbf{\Omega}_3\}} {}_1F_1(a + b + c; c; \mathbf{\Omega}_3 \mathbf{N})$$

and $\mathbf{F}_1 \sim \mathcal{B}II(A)_m(a, c, \mathbf{\Omega}_3)$ and $\mathbf{F}_2 \sim \mathcal{B}II(A)_m(b, c, \mathbf{\Omega}_3)$.

2. If $\Omega_3 = \mathbf{0}$

$$\frac{\mathcal{BG}BII_{2m \times m}(\mathbb{F}; a, b, c)}{\text{etr}\{\Omega_1 + \Omega_2\}} \sum_{\kappa, \tau; \phi}^{\infty} \frac{(a+b+c)_{\phi} \theta_{\phi}^{\kappa, \tau}}{(a)_{\kappa} (b)_{\tau} k! l!} C_{\phi}^{\kappa, \tau}(\Omega_1 \mathbf{N}_1, \Omega_2 \mathbf{N}_2).$$

In addition, we have that $\mathbf{F}_1 \sim BII(B)_m(a, c, \Omega_1)$ and $\mathbf{F}_2 \sim BII(B)_m(b, c, \Omega_2)$.

3. If $\Omega_2 = \Omega_3 = \mathbf{0}$

$$\frac{\mathcal{BG}BII_{2m \times m}(\mathbb{F}; a, b, c)}{\text{etr}\{\Omega_1\}} {}_1F_1(a+b+c; c; \Omega_1 \mathbf{N}_1)$$

and with $\mathbf{F}_1 \sim BII(B)_m(a, c, \Omega_1)$ and $\mathbf{F}_2 \sim BII_m(b, c)$.

4. If $\Omega_1 = \Omega_3 = \mathbf{0}$

$$\frac{\mathcal{BG}BII_{2m \times m}(\mathbb{F}; a, b, c)}{\text{etr}\{\Omega_2\}} {}_1F_1(a+b+c; c; \Omega_2 \mathbf{N}_2)$$

and $\mathbf{F}_1 \sim BII_m(a, c)$ and $\mathbf{F}_2 \sim BII(B)_m(b, c, \Omega_2)$.

5. If $\Omega_2 = \mathbf{0}$

$$\frac{\mathcal{BG}BII_{2m \times m}(\mathbb{F}; a, b, c)}{\text{etr}\{\Omega_1 + \Omega_3\}} \sum_{\kappa, \lambda; \phi}^{\infty} \frac{(a+b+c)_{\phi} \theta_{\phi}^{\kappa, \lambda}}{(a)_{\kappa} (c)_{\lambda} k! l!} C_{\phi}^{\kappa, \lambda}(\Omega_1 \mathbf{N}_1, \Omega_3 \mathbf{N}).$$

In addition note that $\mathbf{F}_1 \sim BII_m(a, c, \Omega_1, \Omega_3)$ and $\mathbf{F}_2 \sim BII(A)_m(b, c, \Omega_3)$.

6. If $\Omega_1 = \mathbf{0}$

$$\frac{\mathcal{BG}BII_{2m \times m}(\mathbb{F}; a, b, c)}{\text{etr}\{\Omega_2 + \Omega_3\}} \sum_{\tau, \lambda; \phi}^{\infty} \frac{(a+b+c)_{\phi} \theta_{\phi}^{\tau, \lambda}}{(b)_{\tau} (c)_{\lambda} t! l!} C_{\phi}^{\tau, \lambda}(\Omega_2 \mathbf{N}_2, \Omega_3 \mathbf{N})$$

and where $\mathbf{F}_1 \sim BII(A)_m(a, c, \Omega_3)$ and $\mathbf{F}_2 \sim BII_m(b, c, \Omega_2, \Omega_3)$.

Proof. The proof is obtained in a similar way to the proof of Corollary 3.1. \square

5 Conclusions

The problem to finding the density function of a doubly noncentral beta type II distribution has been studied by different authors, see Davis (1979), Chikuse (1980) and Díaz-García and Gutiérrez-Jáimez (2006), among others. All these authors, in fact, found the symmetrised density function of a doubly noncentral beta type II distribution. The nonsymmetrised density function remained an open problem. The Theorem 2.1 solves this problem for doubly noncentral beta type I and II distributions by applying in an inverse way the definition of symmetrised function proposed by Greenacre (1973). This approach was used previously by Díaz-García and Gutiérrez-Jáimez (2007) in the noncentral cases.

In a similar way, we have found the symmetrised doubly noncentral generalised beta type I and II distributions and, by applying in an inverse way the definition of symmetrised function proposed by Greenacre (1973), we have found the nonsymmetrised doubly noncentral generalised beta type I and II distributions. Finally, as corollaries, we studied all the different noncentral generalised beta type I and II distributions.

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